Journal of Pure and Applied Mathematics: Advances and Applications
Volume 2, Number 2, 2009, Pages 169-185

BACKWARD EULER METHOD FOR THE DIFFUSION EQUATION WITH INTEGRAL BOUNDARY SPECIFICATIONS

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Abstract

This paper is considered to solve diffusion equation with integral boundary specifications model various physical problems. We used backward Euler method. In the end, we find the approximation solution. The new algorithms are tested on two problems from the literature.

1. Introduction

This paper is concerned with the numerical solution of the diffusion equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad 0 < x < 1, \quad 0 < t \leq T, \tag{1.1}$$

2000 Mathematics Subject Classification: 35K05, 35K20, 65N06.

Keywords and phrases: diffusion equation, integral boundary specifications, backward Euler method.

Received June 10, 2009; Revised September 28, 2009

subject to the initial condition

$$u(x, 0) = \varphi(x), \quad 0 < x < 1, \tag{1.2}$$

and the integral boundary conditions

$$u(0, t) = \int_0^1 p(x, t)u(x, t)dx + E(t), \quad 0 < t \leq T, \tag{1.3}$$

$$u(1, t) = \int_0^1 q(x, t)u(x, t)dx + G(t), \quad 0 < t \leq T, \tag{1.4}$$

where $f, \varphi, p, q, G,$ and E are known functions, so we must be determined the function u.

Problem of the form (1.1)-(1.4) arise, e.g., in the quasistatic theory of thermoelasticity treated by several mathematicians such as Day [2], [3], under the assumption that

$$\int_0^1 |p(x)|dx < 1, \quad \int_0^1 |q(x)|dx < 1. \tag{1.5}$$

The numerical solution of problem (1.1)-(1.4) was considered by a number of authors, see, e.g., [1, 4-10].

This paper is outlined as follows: Section 2 is devoted to the backward Euler method. Section 3 contains some numerical results, and conclusion in the last section.

2. Backward Euler Method

First, we take positive integers N and M. We divide the intervals $[0, 1]$ and $[0, T]$ into M and N subintervals of equal lengths $\delta x = 1 / M$ and $\delta t = T / N,$ respectively. By u_i^n, we denote the approximation to u at the i-th grid-point and n-th time step. We also introduce the following notation:

$$\delta_x^2 u_i^n = u_{i-1}^n - 2u_i^n + u_{i+1}^n.$$

The Grid point (x_i, t_n) are given by

$$x_i = i(\delta x), \quad i = 0, 1, 2, \dots, M,$$

$$t_n = n(\delta t), \quad n = 0, 1, 2, \dots, N, \tag{2.1}$$

where M is an even integer.

Using the classical backward Euler method to approximate the derivative (1.1), we have

$$\frac{u_i^{n+1} - u_i^n}{\delta t} - \alpha\left(\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{(\delta x)^2}\right) = f_i^{n+1}, \tag{2.2}$$

after some rearrangement, the Equation (2.3) becomes

$$-ru_{i-1}^{n+1} + (1 + 2r)u_i^{n+1} - ru_{i+1}^{n+1} = u_i^n + (\delta t)f_i^{n+1}, \tag{2.3}$$

where

$$r \approx \frac{\alpha\delta t}{(\delta x)^2}. \tag{2.4}$$

There are $M - 1$ linear equations from (2.3) in $M + 1$ unknowns $u_0, u_1, \cdots, u_M$. In order to solve for unknowns, we need two more equations. So, let us formally approximate the integrals in (1.3) and (1.4) numerically by Simpson's rule (which requires M to be even). Letting $M = 2m$, yields

$$\begin{aligned} u_0^k &= \int_0^1 p(x, t)u(x, t)dx \\ &= \frac{(\delta x)}{3}\left(p_0^k u_0^k + 4\sum_{i=1}^{m} p_{2i-1}^k u_{2i-1}^k + 2\sum_{i=1}^{m-1} p_{2i}^k u_{2i}^k + p_M^k u_M^k\right) \\ &\quad + E^k + O(\delta x^4), \end{aligned}$$

$$u_M^k = \int_0^1 q(x, t)u(x, t)dx$$

$$= \frac{(\delta x)}{3}\left(q_0^k u_0^k + 4\sum_{i=1}^{m} q_{2i-1}^k u_{2i-1}^k + 2\sum_{i=1}^{m-1} q_{2i}^k u_{2i}^k + q_M^k u_M^k\right)$$

$$+ G^k + O(\delta x^4).$$

Thus, we can write

$$\big((\delta x)p_0^{n+1} - 3\big)u_0^{n+1} + 4(\delta x)p_1^{n+1}u_1^{n+1} + 2(\delta x)p_2^{n+1}u_2^{n+1}$$
$$+ \cdots + 4(\delta x)p_{M-1}^{n+1}u_{M-1}^{n+1} + (\delta x)p_M^{n+1}u_M^{n+1} \simeq -3E^{n+1}, \qquad (2.5)$$

and

$$(\delta x)q_0^{n+1}u_0^{n+1} + 4(\delta x)q_1^{n+1}u_1^{n+1} + 2(\delta x)q_2^{n+1}u_2^{n+1}$$
$$+ \cdots + 4(\delta x)q_{M-1}^{n+1}u_{M-1}^{n+1} + \big((\delta x)q_M^{n+1} - 3\big)u_M^{n+1} \simeq -3G^{n+1}. \qquad (2.6)$$

We write the system in the matrix form

$$A^{n+1}U^{n+1} = B^{n+1},$$

where

$$A^{n+1} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & & & a_{0(M-2)} & a_{0(M-1)} & a_{0M} \\ -r & 1+2r & -r & 0 & \cdots & & & 0 \\ 0 & -r & 1+2r & -r & 0 & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & & 0 & -r & 1+2r & -r \\ a_{M0} & a_{M1} & a_{M2} & & & a_{M(M-2)} & a_{M(M-1)} & a_{MM} \end{pmatrix},$$

(2.7)

$$B^{n+1} = \begin{pmatrix} -3E^{n+1} \\ u_1^n + (\delta t)f_1^n \\ \vdots \\ u_{M-1}^n + (\delta t)f_{M-1}^n \\ -3G^{n+1} \end{pmatrix}, \qquad (2.8)$$

where

$$a_{00} = (\delta x)p_0^{n+1} - 3,$$

$$a_{0,2i-1} = 4(\delta x)p_{2i-1}^{n+1}, \quad i = 1 : \frac{M}{2},$$

$$a_{0,2i} = 2(\delta x)p_{2i}^{n+1}, \quad i = 1 : \frac{M}{2} - 1,$$

$$a_{0M} = (\delta x)p_M^{n+1},$$

$$a_{M0} = (\delta x)q_0^{n+1},$$

$$a_{0,2i-1} = 4(\delta x)p_{2i-1}^{n+1}, \quad i = 1 : \frac{M}{2},$$

$$a_{0,2i} = 2(\delta x)p_{2i}^{n+1}, \quad i = 1 : \frac{M}{2} - 1,$$

and

$$a_{MM} = (\delta x)q_M^{n+1} - 3.$$

2.1. Numerical tests. To test the above algorithm, we use two examples as follows:

Test 1. Consider (1.1)-(1.4) with

$$f(x, t) = -x(x-1)\exp(-t), \tag{2.9}$$

$$\varphi(x) = x(x-1) - 2, \tag{2.10}$$

$$p(x) = \frac{12}{13}, \quad q(x) = \frac{12}{13}, \tag{2.11}$$

$$G(t) = 0, \quad E(t) = 0. \tag{2.12}$$

It is easy to check that the exact solution of this test problem is

$$u(x, t) = (x(x-1) - 2)\exp(-t). \tag{2.13}$$

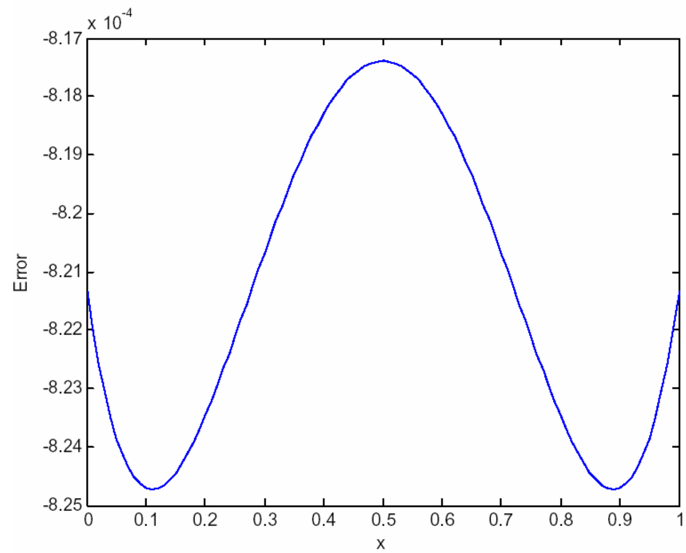

Figure 1. $T = 0.2.$

Table 1. $T = 0.2,\ N = M = 100$

x	$Error$	$U_{approx.}$	U_{exact}
0.10	−0.00082471	−1.71255848	−1.71114727
0.20	−0.00082346	−1.76991469	−1.76845843
0.30	−0.00082068	−1.81087990	−1.80939496
0.40	−0.00081830	−1.83545761	−1.83395689
0.50	−0.00081739	−1.84364995	−1.84214419
0.60	−0.00081830	−1.83545761	−1.83395689
0.70	−0.00082068	−1.81087990	−1.80939496
0.80	−0.00082346	−1.76991469	−1.76845843
0.90	−0.00082471	−1.71255848	−1.71114727
1.00	−0.00082135	−1.63880643	−1.63746151

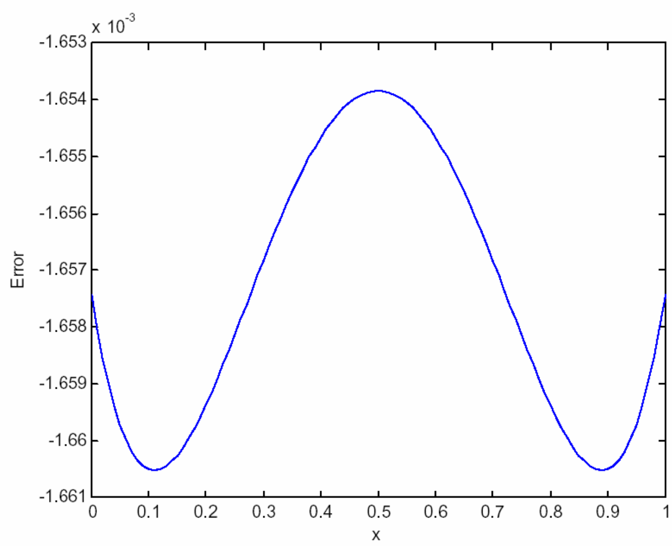

Figure 2. $T = 0.4$.

Table 2. $T = 0.4$, $N = M = 100$

x	*Error*	$U_{approx.}$	U_{exact}
0.10	–0.00166050	–1.40329520	–1.40096890
0.20	–0.00165936	–1.45029388	–1.44789130
0.30	–0.00165683	–1.48386174	–1.48140730
0.40	–0.00165466	–1.50400141	–1.50151690
0.50	–0.00165384	–1.51071445	–1.50822010
0.60	–0.00165466	–1.50400141	–1.50151690
0.70	–0.00165683	–1.48386174	–1.48140730
0.80	–0.00165936	–1.45029388	–1.44789130
0.90	–0.00166050	–1.40329520	–1.40096890
1.00	–0.00165744	–1.34286212	–1.34064009

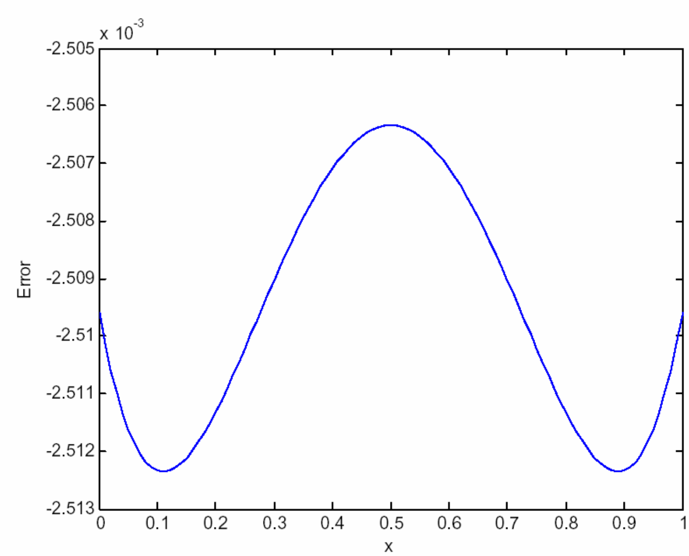

Figure 3. $T = 0.6.$

Table 3. $T = 0.6,\ N = M = 100$

x	$Error$	$U_{approx.}$	U_{exact}
0.10	−0.00251234	−1.14989801	−1.14701632
0.20	−0.00251132	−1.18841013	−1.18543313
0.30	−0.00250904	−1.21591686	−1.21287372
0.40	−0.00250709	−1.23242012	−1.22933806
0.50	−0.00250635	−1.23792108	−1.23482618
0.60	−0.00250709	−1.23242012	−1.22933806
0.70	−0.00250904	−1.21591686	−1.21287372
0.80	−0.00251132	−1.18841013	−1.18543313
0.90	−0.00251234	−1.14989801	−1.14701632
1.00	−0.00250959	−1.10037785	−1.09762327

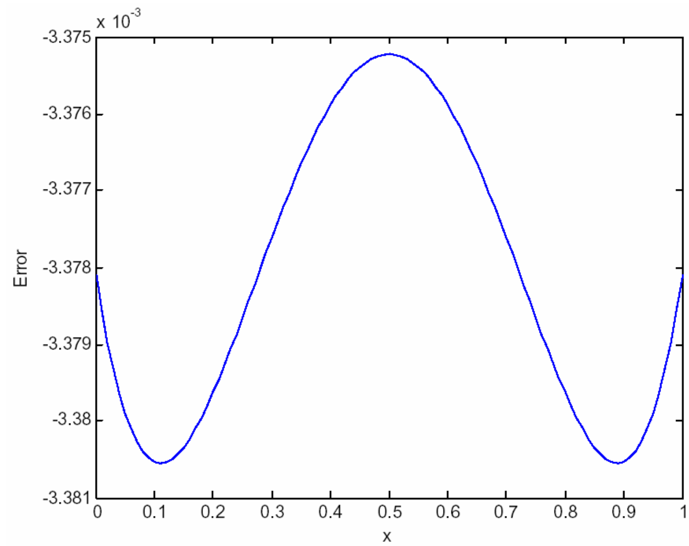

Figure 4. $T = 0.8$.

Table 4. $T = 0.8$, $N = M = 100$

x	*Error*	$U_{approx.}$	U_{exact}
0.10	−0.00338053	−0.94227219	−0.93909754
0.20	−0.00337963	−0.97383066	−0.97055056
0.30	−0.00337761	−0.99637104	−0.99301701
0.40	−0.00337589	−1.00989470	−1.00649688
0.50	−0.00337523	−1.01440249	−1.01099017
0.60	−0.00337589	−1.00989470	−1.00649688
0.70	−0.00337761	−0.99637104	−0.99301701
0.80	−0.00337963	−0.97383066	−0.97055056
0.90	−0.00338053	−0.94227219	−0.93909754
1.00	−0.00337810	−0.90169368	−0.89865793

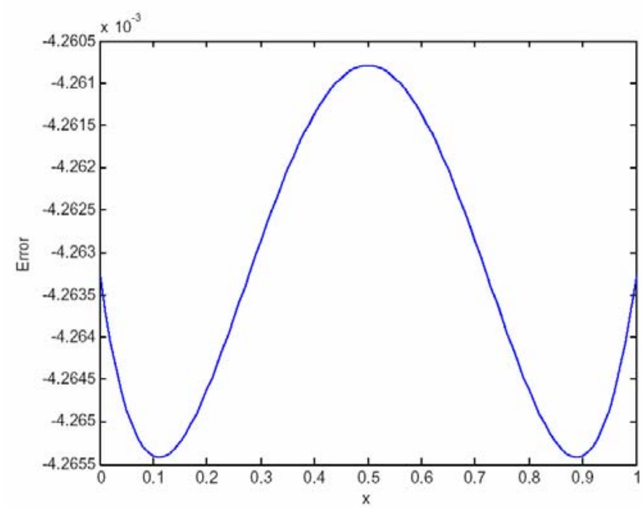

Figure 5. $T = 1.0$.

Table 5. $T = 1.0$, $N = M = 100$

x	$Error$	$U_{approx.}$	U_{exact}
0.10	−0.00426540	−0.77214756	−0.76886803
0.20	−0.00426462	−0.79800834	−0.79461959
0.30	−0.00426287	−0.81647933	−0.81301356
0.40	−0.00426137	−0.82756153	−0.82404995
0.50	−0.00426079	−0.83125552	−0.82772874
0.60	−0.00426137	−0.82756153	−0.82404995
0.70	−0.00426287	−0.81647933	−0.81301356
0.80	−0.00426462	−0.79800834	−0.79461959
0.90	−0.00426540	−0.77214756	−0.76886803
1.00	−0.00426329	−0.73889563	−0.73575888

Test 2. Now, consider problem (1.1)-(1.4) with

$$f(x, t) = -\exp(-t)\left(x(x-1) + \frac{\delta}{6(1+\delta)} + 2\right),$$

$$x \in (0, 1),\ 0 < t \leq T, \tag{2.14}$$

$$\varphi(x) = x(x-1) + \frac{\delta}{6(1+\delta)}, \quad x \in [0, 1], \tag{2.15}$$

$$p(x) = q(x) = -\delta, \tag{2.16}$$

and

$$G(t) = 0, \quad E(t) = 0, \tag{2.17}$$

where $\delta = 0.0144$, and the exact solution is

$$u(x, t) = \exp(-t)\left(x(x-1) + \frac{\delta}{6(1+\delta)}\right). \tag{2.18}$$

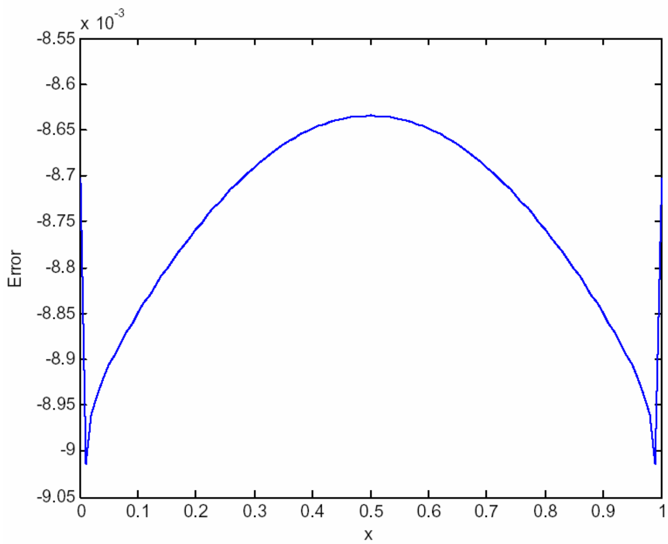

Figure 6. $T = 0.2$.

Table 6. $T = 0.2$, $N = M = 100$

x	$Error$	$U_{approx.}$	U_{exact}
0.10	−0.00884950	−0.07238365	−0.07174871
0.20	−0.00875833	−0.13019021	−0.12905986
0.30	−0.00869080	−0.17147380	−0.16999640
0.40	−0.00864896	−0.19624105	−0.19455832
0.50	−0.00863479	−0.20449629	−0.20274563
0.60	−0.00864896	−0.19624105	−0.19455832
0.70	−0.00869080	−0.17147380	−0.16999640
0.80	−0.00875833	−0.13019021	−0.12905986
0.90	−0.00884950	−0.07238365	−0.07174871
1.00	−0.00870274	0.00195392	0.00193706

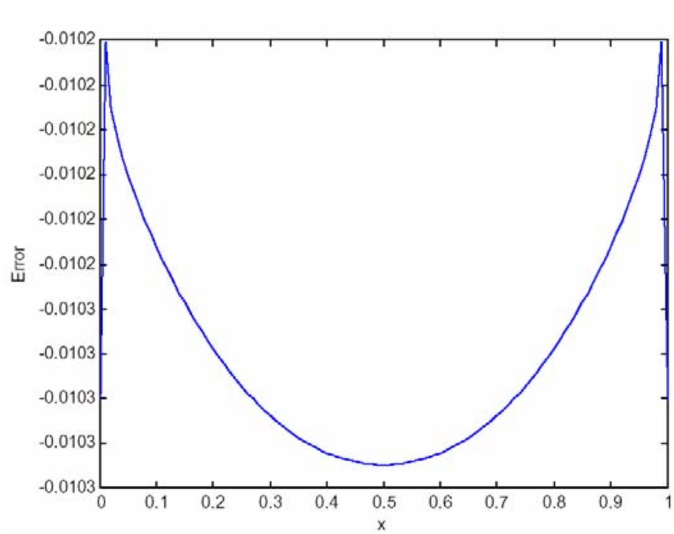

Figure 7. $T = 0.4$.

Table 7. $T = 0.4,\ N = M = 100$

x	$Error$	$U_{approx.}$	U_{exact}
0.10	–0.01024310	–0.05934458	–0.05874287
0.20	–0.01025449	–0.10674882	–0.10566528
0.30	–0.01026189	–0.14060954	–0.13918128
0.40	–0.01026606	–0.16092617	–0.15929088
0.50	–0.01026741	–0.16769841	–0.16599408
0.60	–0.01026606	–0.16092617	–0.15929088
0.70	–0.01026189	–0.14060954	–0.13918128
0.80	–0.01025449	–0.10674882	–0.10566528
0.90	–0.01024310	–0.05934458	–0.05874287
1.00	–0.01026017	0.00160220	0.00158593

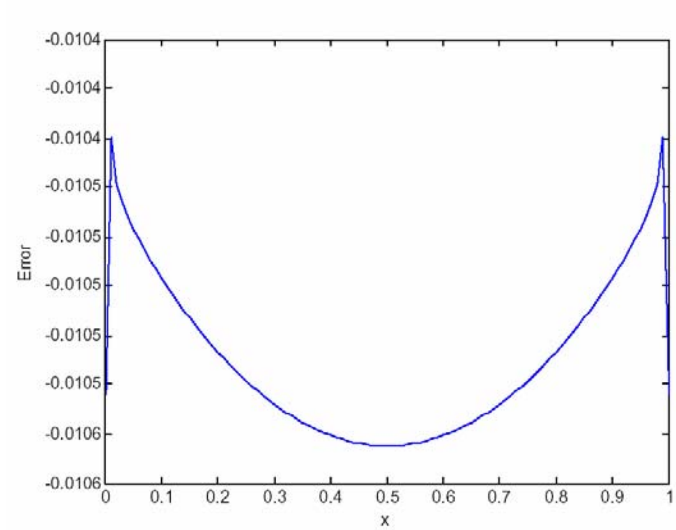

Figure 8. $T = 0.6$.

Table 8. $T = 0.6,\ N = M = 100$

x	$Error$	$U_{approx.}$	U_{exact}
0.10	–0.01049687	–0.04859944	–0.04809460
0.20	–0.01052693	–0.08742211	–0.08651141
0.30	–0.01054797	–0.11515396	–0.11395199
0.40	–0.01056053	–0.13179361	–0.13041634
0.50	–0.01056470	–0.13734025	–0.13590446
0.60	–0.01056053	–0.13179361	–0.13041634
0.70	–0.01054797	–0.11515396	–0.11395199
0.80	–0.01052693	–0.08742211	–0.08651141
0.90	–0.01049687	–0.04859944	–0.04809460
1.00	–0.01054377	0.00131214	0.00129845

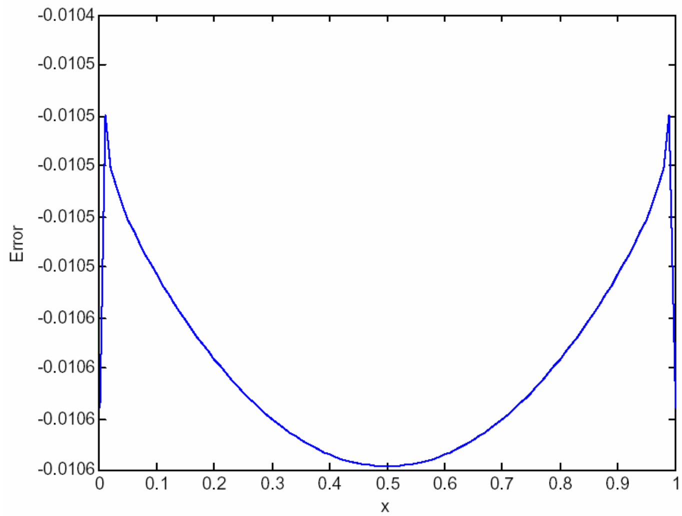

Figure 9. $T = 0.8$.

Table 9. $T = 0.8$, $N = M = 100$

x	*Error*	$U_{approx.}$	U_{exact}
0.10	–0.01054308	–0.03979168	–0.03937653
0.20	–0.01057654	–0.07157868	–0.07082955
0.30	–0.01060007	–0.09428495	–0.09329600
0.40	–0.01061415	–0.10790920	–0.10677587
0.50	–0.01061884	–0.11245071	–0.11126916
0.60	–0.01061415	–0.10790920	–0.10677587
0.70	–0.01060007	–0.09428495	–0.09329600
0.80	–0.01057654	–0.07157868	–0.07082955
0.90	–0.01054308	–0.03979168	–0.03937653
1.00	–0.01059541	0.00107434	0.00106308

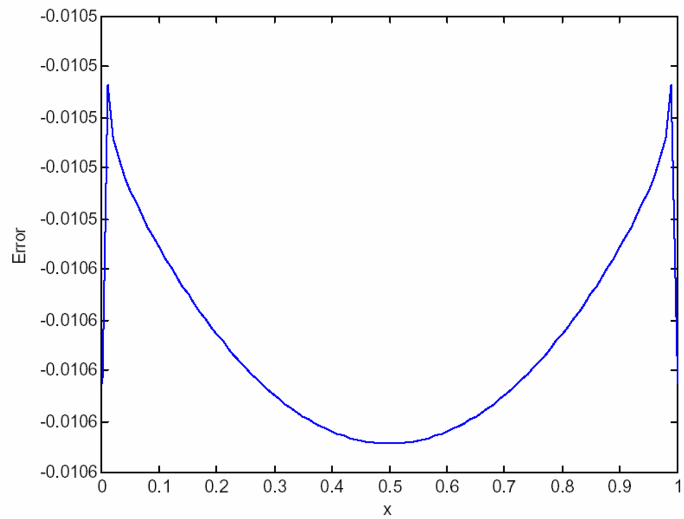

Figure 10. $T = 1.0$.

Table 10. $T = 1.0$, $N = M = 100$

x	*Error*	$U_{approx.}$	U_{exact}
0.10	–0.01055149	–0.03257894	–0.03223877
0.20	–0.01058558	–0.05860419	–0.05799033
0.30	–0.01060955	–0.07719471	–0.07638431
0.40	–0.01062391	–0.08834944	–0.08742069
0.50	–0.01062869	–0.09206775	–0.09109948
0.60	–0.01062391	–0.08834944	–0.08742069
0.70	–0.01060955	–0.07719471	–0.07638431
0.80	–0.01058558	–0.05860419	–0.05799033
0.90	–0.01055149	–0.03257894	–0.03223877
1.00	–0.01060481	0.00087961	0.00087038

3. Conclusion

We have presented a backward Euler method for one-dimensional diffusion equation with integral boundary specifications. It is scheme even more implicit: an implicit scheme plus implicit boundary conditions. Numerical examples are provided to confirm the accuracy.

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